

# A Dynkin Diagram Classification Theorem Arising from a Combinatorial Problem\*

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A combinatorial-linear algebraic condition sufficient for a ranked partially ordered set to be rank unimodal and strongly Sperner is presented. The distributive lattices which satisfy this condition are classified. These lattices are indexed by Dynkin diagrams of type **ADE**, which actually appear embedded in the Hasse diagrams of the lattices. © 1986 Academic Press, Inc.

## 1. INTRODUCTION AND RESULTS

In this paper we will classify which distributive lattices satisfy a certain combinatorial-linear algebraic condition which is sufficient for the strong Sperner property. The resulting distributive lattices are indexed by a certain family of Dynkin-like diagrams which occur elsewhere in mathematics. Each Dynkin diagram actually occurs in the lower part of the Hasse diagram of the lattice which it indexes.

A *ranked poset*  $L$  is a partially ordered set  $L$  together with a partition  $L = \bigcup_{i=0}^r L_i$  such that elements in *rank*  $L_i$  cover only elements in *rank*  $L_{i-1}$ . It is *rank symmetric* if  $|L_i| = |L_{r-i}|$  and *rank unimodal* if there is some  $m$  such that  $|L_0| \leq |L_1| \leq \cdots \leq |L_m| \geq |L_{m+1}| \geq \cdots \geq |L_r|$ . It is *strongly Sperner* if for every  $k \geq 1$  the largest union of  $k$  antichains is no larger than the largest union of  $k$  ranks. The Sperner ( $k=1$  only) and strong Sperner properties have been studied for several years by various people, see, e.g. [G-K] or [GSS].

Associate to any ranked poset

$$L = \bigcup_{i=0}^r L_i$$

\* Contained in the author's doctoral thesis written under the direction of R. P. Stanley at M.I.T., 1981.

a graded complex vector space

$$\tilde{L} = \bigoplus_{i=0}^r \tilde{L}_i,$$

where  $\tilde{L}_i$  is the complex vector space freely generated by vectors  $\tilde{a}$  corresponding to elements of  $L_i$ . A linear operator  $X$  on  $\tilde{L}$  is a *lowering operator* if  $X\tilde{L}_i \subseteq \tilde{L}_{i-1}$ . It is a *raising operator* if  $X\tilde{L}_i \subseteq \tilde{L}_{i+1}$ . A raising operator defined by

$$X\tilde{a} = \sum \theta(a, b) \tilde{b}$$

is an *order raising operator* if  $\theta(a, b) \neq 0$  implies  $b$  covers  $a$  in  $L$ . Define a linear operator  $H$  on  $\tilde{L}$  by

$$H\tilde{L}_i = (2i - r) \tilde{L}_i.$$

The poset  $L$  carries a representation of  $\mathfrak{sl}(2, \mathbb{C})$  if there exist a lowering operator  $Y$  and an order raising operator  $X$  on  $\tilde{L}$  such that  $XY - YX = H$ .

The following proposition is the main result of [Pr1]; it incorporates the combinatorial-linear algebraic technique Lemma 1.1 of [St1].

**PROPOSITION 1.** *A ranked poset is rank symmetric, rank unimodal, and strongly Sperner if and only if it carries a representation of  $\mathfrak{sl}(2, \mathbb{C})$ .*

It can be shown that this proposition can be applied to any Bruhat poset arising from a Weyl group [Pr1, p. 278; St1]. This proposition was also used to give a short proof of the fact the product of two rank symmetric rank unimodal strongly Sperner posets also has these properties [Pr1]. (There is no known combinatorial proof of this fact; it was first proved using linear algebra in [Can; PSS].) Finally, this proposition can be used to give a short proof of the main result of [Har; St2]: Suppose a group  $G$  acts on the ranks of a poset  $P$  which carries a representation of  $\mathfrak{sl}(2, \mathbb{C})$ . If the action of  $G$  commutes with the operator  $X$ , then the quotient poset  $P/G$  is rank symmetric, rank unimodal, and strongly Sperner.

Application of Proposition 1 to an arbitrary ranked poset requires the solution of  $|L|$  simultaneous quadratic equations in the coefficients of the  $X$  and  $Y$  operators. These equations become linear if all of the coefficients of the  $X$  operator are set equal to unity. The equations assume a particularly nice form if  $L$  is a “uniquely modular” poset. (Definition:  $L$  is *uniquely modular* if whenever two elements cover a third element there exists a unique fourth element which covers the first two elements; and order dually.) Since the equations at hand are now linear, we can assume without loss of generality that the coefficients are rational numbers.

DEFINITION. Let  $L$  be a uniquely modular poset with  $r + 1$  ranks. Then  $L$  is *edge-labelable* if each edge of the Hasse diagram for  $L$  can be labeled with a rational number such that

(i) opposite edges of any square in the Hasse diagram have equal labels;

(ii) if  $b \in L_i$ , then the sum of the labels on edges emanating below  $b$  minus the sum of the labels on edges emanating above  $b$  equals  $2i - r$ .

Using the edge labels as the coefficients for the  $Y$  operator in Proposition 1, one obtains

PROPOSITION 2. *Edge-Labelable lattices are rank symmetric, rank unimodal, and strongly Sperner.*

A subset  $I$  of a poset  $P$  is an order ideal of  $P$  if  $y \in I$  and  $x \leq y$  imply  $x \in I$ . The poset  $J(P)$  of all order ideals of  $P$  is always a distributive lattice. Conversely, for any distributive lattice  $L$  there is a unique poset  $P$ , the poset of join irreducibles of  $L$ , such that  $L = J(P)$ . If  $s \geq 1$ , let  $s$  denote the totally ordered set with  $s$  elements.

The Bruhat posets which are distributive lattices were shown to be edge-labelable in Sections 10 and 12 of [Pr2] by composing a minuscule representation of a simple Lie algebra with the embedding of a principal 3-dimensional subalgebra. Surprisingly, it is possible to prove that these are the only distributive lattices which can be edge-labeled.

THEOREM 1. *The only edge-labelable distributive lattices are  $J(\mathbf{s} \times \mathbf{t})$ ,  $s, t \geq 1$ ,  $J^2(\mathbf{2} \times \mathbf{t})$ ,  $t \geq 1$ ,  $J^k(\mathbf{2} \times \mathbf{2})$ ,  $k \geq 1$ ,  $J^3(\mathbf{2} \times \mathbf{3})$ ,  $J^4(\mathbf{2} \times \mathbf{3})$ , and products of these lattices.*

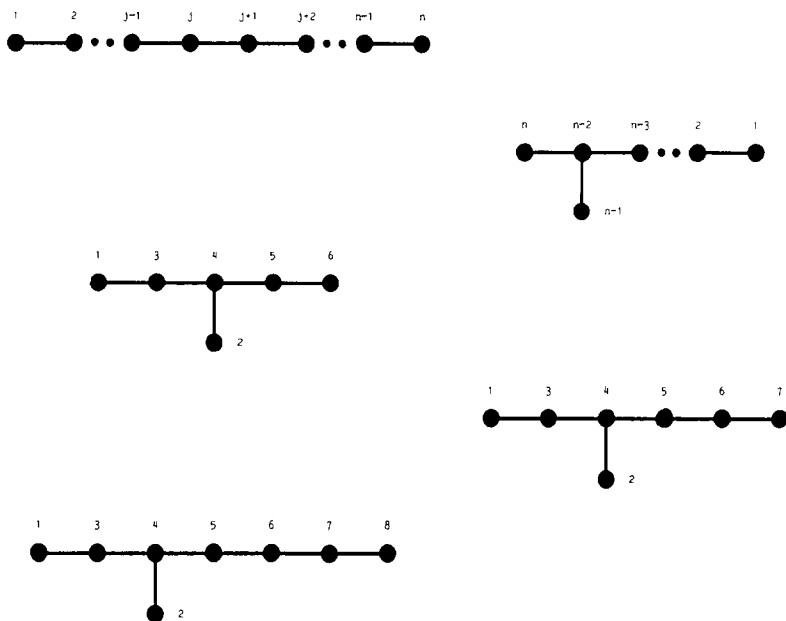
The edge labels for these lattices will be explicitly computed during the proof of Theorem 2.

If we restrict our attention to uniquely modular posets which are distributive lattices, then the edge-labelable condition can be stated more elegantly in terms of the poset of join irreducibles.

DEFINITION. A poset  $P$  is *vertex-labelable* if there exists a function  $\pi: P \rightarrow \mathbb{Q}$  such that the equations

$$\sum_{\substack{x \text{ maximal} \\ \text{in } I}} \pi(x) - |I| = \sum_{\substack{y \text{ minimal} \\ \text{in } P - I}} \pi(y) - |P - I|$$

are satisfied for every order ideal  $I \subseteq P$ .

FIG. 1. Dynkin diagrams of type **ADE**.

**PROPOSITION 3.** *A poset  $P$  is vertex-labelable if and only if the distributive lattice  $L = J(P)$  is edge-labelable.*

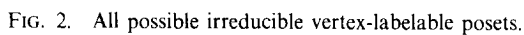
A connected Dynkin diagram of type **ADE** with special node  $j$ , denoted  $\mathbf{X}_n[j]$ ,  $\mathbf{X} \in \{\mathbf{A}, \mathbf{D}, \mathbf{E}\}$ ,  $1 \leq j \leq n$ , is one of the diagrams of Fig. 1 with one of its nodes designated as special.

A poset is *irreducible* if it cannot be expressed as the direct sum (disjoint union) of two non-empty posets. The *basic tree* of an irreducible poset  $P$  is the multi-rooted tree (connected acyclic graph with special vertices) whose vertices are the elements  $x$  in  $P$  such that  $\{y: y \leq x\}$  is a chain, whose edges are the covering relations between these vertices, and whose roots (special vertices) are the minimal elements of  $P$ .

We can now state the main result of this paper in full detail:

**THEOREM 2.** *The only irreducible vertex-labelable posets are  $\mathbf{s} \times \mathbf{t}$ ,  $s, t \geq 1$ ,  $J(\mathbf{2} \times \mathbf{t})$ ,  $t \geq 1$ ,  $J^k(\mathbf{2} \times \mathbf{2})$ ,  $k \geq 0$ ,  $J^2(\mathbf{2} \times \mathbf{3})$ , and  $J^3(\mathbf{2} \times \mathbf{3})$ . The basic trees of these posets are, respectively,  $\mathbf{A}_{s+t-1}[s]$ ,  $\mathbf{D}_{t+2}[t+2]$ ,  $\mathbf{D}_{k+3}[1]$ ,  $\mathbf{E}_6[6]$ , and  $\mathbf{E}_7[7]$ . Arbitrary vertex-labelable posets are direct sums of these posets.*

The irreducible vertex-labelable posets are shown in Fig. 2. The elements of the embedded Dynkin diagrams (basic trees) are denoted with solid dots. The proof of Theorem 2 will show that the vertex labelings shown in Fig. 2 are the only possible vertex labelings in each case. Note that the



diagrams  $\mathbf{A}_{s+t-1}[t]$ ,  $\mathbf{D}_{t+2}[t+1]$ , and  $\mathbf{E}_6[1]$  are identical to the diagrams  $\mathbf{A}_{s+t-1}[s]$ ,  $\mathbf{D}_{t+2}[t+2]$ , and  $\mathbf{E}_6[6]$ .

It is not unusual for the Dynkin diagrams of types **A**, **D**, and **E** to receive special attention [HHS]. (These are the Dynkin diagrams without multiple edges, or equivalently, the diagrams that index root systems with all roots of equal length.) Besides indexing minuscule representations of simple Lie algebras of type **ADE** [Hum, Exercise 13.13], the particular diagrams with special node listed above also index the Hermitian symmetric spaces of type ADE [Wol, p. 289]. The distributive lattices listed in Theorem 1 are exactly the Bruhat posets of type **ADE** which are lattices. They arise in representation theory as the sets of weights of minuscule representations of simple Lie algebras. The relationship between the posets of Theorem 2 and the lattices of Theorem 1 can be described in terms of the roots and weights of the corresponding minuscule representation. The vertex labels of the vertex-labelable posets turn out to always be positive integers. These numbers can be interpreted in a geometric context as the coefficients of the Hodge adjoint of cup product multiplication with a hyperplane section in the cohomology ring of a minuscule flag manifold. See Sections 3, 4, 11, and 12 of [Pr2] for elucidation of the preceding remarks.

The most interesting aspect of the proof of Theorem 2 is that a combinatorial consequence of Proposition 2 is used for the key step. A Kirchoff conservation of current-network argument uses the combinatorial fact to prove that the vertex labels must always be positive. This result is then used to prove three lemmas concerning the local structure of vertex-labelable posets. These lemmas greatly reduce the possibilities for the basic trees of irreducible components of vertex-labelable posets. Systems of linear equations closely related to the Cartan matrices of simple Lie algebras are then used to further narrow the possibilities for basic trees to those listed above together with  $\mathbf{E}_6[2]$ ,  $\mathbf{E}_7[1]$ ,  $\mathbf{E}_7[2]$ ,  $\mathbf{E}_8[8]$ ,  $\mathbf{E}_8[1]$ , and  $\mathbf{E}_8[2]$ . Finally, the original definition of vertex-labelable and the local structure lemmas are used to either eliminate a potential basic tree or to directly construct the unique possible irreducible component corresponding to it. The “bad” basic trees listed above correspond to fundamental representations of simple Lie algebras which are not quite minuscule.

Unlike most other Dynkin classification procedures, it is not possible to immediately reduce to the irreducible case in this proof. The following corollary is a consequence of the constructive last part of the proof of the theorem.

**COROLLARY.** *A poset is vertex-labelable if and only if each of its irreducible components is vertex-labelable.*

## 2. PROOFS OF THEOREM 2 AND COROLLARY

From now on,  $P$  denotes a vertex-labelable poset with  $p$  elements and labeling function  $\pi$ . For simplicity of notation, the same symbols  $x, y, \dots$ , are used to refer both to elements of  $P$  and to the vertex labels  $\pi(x), \pi(y), \dots$ . Similarly, an upper case Latin letter can refer to either a subset of  $P$  or to the sum of the vertex labels of the elements in the subset.

The following lemma will be used in five distinct steps later in the proof.

LEMMA 1. *All vertex labels are positive.*

*Proof.* Consider  $L = J(P)$ . This distributive lattice has  $p + 1$  ranks. The Hasse diagram of  $L$  can be viewed as a network, where a vertex in the  $i$ th rank of  $L$  is a source or sink of  $(2i - p)$  units of flow, and an edge corresponding to an element  $x$  in  $P$  carries  $\pi(x)$  units of flow downward. Since  $L$  is edge-labelable, Kirchhoff's first law (conservation of mass) is satisfied at every vertex of  $L$ . Let  $F \subseteq L$  be the complement of any order ideal  $I \subseteq L$ . By the conservation of mass, the sum of the flows on edges passing from  $F$  to  $I$  must equal the sum of the sources and sinks which are members of  $F$ . Vertices in ranks  $L_i$ ,  $i < p/2$ , are sinks. Now  $L$  carries a representation of  $\mathfrak{sl}(2, \mathbb{C})$ . So the proof of Proposition 1 implies that (iii) of Lemma 1.1 of [St1] holds:  $X^{p-2i}: \tilde{L}_i \rightarrow \tilde{L}_{p-i}$  is an isomorphism. Thus (ii) of this lemma holds, viz.  $L$  has "Property  $T$ ": If  $0 \leq i < p/2$ , there exist  $|L_i|$  pairwise disjoint chains from  $L_i$  to  $L_{p-i}$ . (Alternatively, use Prop. 2 and the equivalence: A ranked poset is strong Sperner and rank unimodal iff it has Property  $T$  [GSS].) Therefore, each sink of size  $(2i - p)$  in  $F$  can be matched with a source of size  $-(2i - p)$  which lies in  $F$ . Thus the sum of the sources and sinks in  $F$  is non-negative. In particular, let  $F$  be the set of all order ideals in  $P$  which contain a fixed element  $x$ . Every edge passing from  $F$  to its complement in  $L$  has flow  $\pi(x)$ , and thus the sum of the sources and sinks in  $F$  is a positive integral multiple of  $\pi(x)$ . The sum of sources and sinks in  $F$  is zero only when  $F = L$ , and this  $F$  does not correspond to any poset element  $x$  under the construction above. Therefore,  $\pi(x)$  must be positive.

The following lemma follows immediately from the definition of vertex-labelable.

LEMMA 2. *The poset  $P$  is vertex-labelable if and only if its order dual  $P^*$  is vertex-labelable.*

We will use  $(b, c, \dots)$  to denote the order ideal with maximal elements  $\{b, c, \dots\}$ .

LEMMA 3. *The poset  $P$  is modular; i.e., if element  $b$  and  $c$  both cover  $d$ ,*

then there exists at least one element  $e$  which covers both  $b$  and  $c$ ; and order dually. Hence  $P$  is ranked.

*Proof.* Let  $D = \{d \text{ covered by } b \text{ and } c\}$ ,

$E = \{e \text{ which cover } b \text{ and } c\}$ ,

$F = \{f \text{ covered by } c \text{ but not } b\}$ ,

$G = \{g \text{ covered by } b \text{ but not } c\}$ ,

$S = \{s \text{ which cover } b \text{ but not } c\}$ ,

$T = \{t \text{ which cover } c \text{ but not } b\}$ .

Finally, let  $m = 2|(b, c)| - p$ . Four equations in nine unknowns are obtained by considering the ideals  $(b, c)$ ,  $(b, c) - \{c\}$ ,  $(b, c) - \{b\}$ , and  $(b, c) - \{b, c\}$ :

$$\begin{array}{rclcl} b+c & -E & -S-T & =m, \\ b-c & +F & -S & =m-2, \\ -b+c & +G & -T & =m-2, \\ -b-c+D & +F+G & & =m-4. \end{array}$$

Solving these equations yields  $E=D$ . Lemma 1 implies  $D > 0$ . Hence  $E$  is non-empty. Use Lemma 2 to obtain the dual result. Apply Theorem II.16 of [Bir] to conclude that  $P$  is ranked.

**LEMMA 4.** *No element ever covers or is covered by three or more other elements.*

*Proof.* Proceed by induction on the ranks of  $P$ . Let  $q$  be an element of minimal rank which covers at least three elements  $b$ ,  $c$ , and  $d$ . Let  $K$  be the set of other elements covered by  $q$ . Figure 3 shows the four possible situations for the highest three ranks of the ideal  $(q)$ . It will become clear

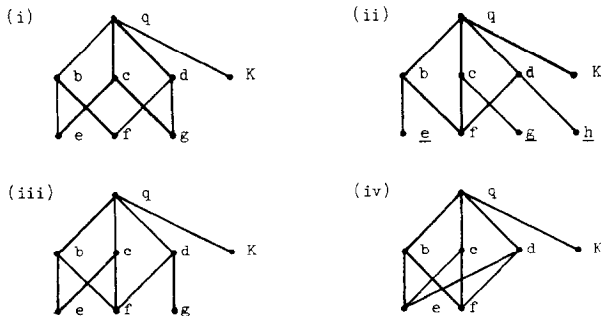


FIG. 3. Lemma 4.



that the existence of the underscored elements is irrelevant. Assume for now that they exist. It will also become clear that covering relations between elements of  $K$  and any of the elements shown in the lowest rank do not affect the outcome. Ignore any such covering relations.

For each case, consider the 8 equations in 17 or 18 unknowns generated by the ideals  $(q) - \{q\}$ ,  $(q) - \{q, b\}$ ,  $(q) - \{q, c\}$ ,  $(q) - \{q, d\}$ ,  $(q) - \{q, b, c\}$ ,  $(q) - \{q, b, d\}$ ,  $(q) - \{q, c, d\}$ , and  $(q) - \{q, b, c, d\}$ . We write out the equations only for case (i); the other cases are similar. Let  $Y$  denote the minimal elements of  $(q) - \{q, b, c, d\}$ , let  $X$  denote the elements which cover  $b$  but not  $c$  or  $d$ , let  $U$  denote the elements which cover  $b$  and  $c$  but not  $d$ , etc. Finally, let  $R$  denote the elements other than  $q$  which cover  $b$ ,  $c$ , and  $d$ , and let  $m = 2|(q)| - p$ . Then

$$\begin{array}{rclclclcl}
 b+c+d & & +K-q-R-S-T-U-V-W-X-Y & = & m-2, \\
 -b+c+d & & +K & -S & -V-W & -Y & = & m-4, \\
 b-c+d & & +K & -T & -V & -X-Y & = & m-4, \\
 b+c-d & & +K & -U & & -Y & = & m-4, \\
 -b-c+d+e & & +K & & -V & -Y & = & m-6, \\
 -b+c-d & +f & +K & & -W & -Y & = & m-6, \\
 b-c-d & +g & +K & & & -X-Y & = & m-6, \\
 -b-c-d+e+f+g & +K & & & & -Y & = & m-8.
 \end{array}$$

Add the 2nd, 3rd, 4th, and 8th equations, and then subtract the 1st, 5th, and 7th equations. The resulting equation is  $q + R = 0$ . For cases (ii) and (iii), the resulting equation is  $f + q + R = 0$ . In case (iv), it is  $e + f + q + R = 0$ . Apply Lemma 1 to obtain contradictions in all cases. Q.E.D.

The next lemma completes the analysis of the local structure of  $P$ .

**LEMMA 5.** *No two elements both cover each of two other elements. Therefore  $P$  is uniquely modular.*

*Proof.* (see Fig. 4.). Suppose that  $d$  and  $e$  both cover  $b$  and  $c$ . Let  $G$  denote the elements in the rank of  $d$  and  $e$  beside these two elements, and

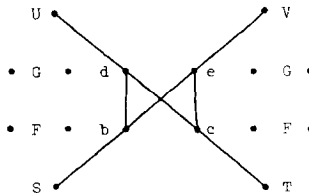


FIG. 4. Lemma 5.

similarly for  $F$ . Let  $S$  ( $T$ ) be the set of elements covered only by  $(b)$ , and let  $U$  ( $V$ ) be the set of elements covering only  $d$  ( $e$ ). Finally, let  $m = 2k - p$ , where  $k$  is the number of elements of  $P$  of rank less than or equal to the rank of  $b$  and  $c$ . Lemma 4 guarantees that the situation described in Fig. 4 is sufficiently general. Consider the ideals  $(d, F)$ ,  $(e, F)$ ,  $(b, F)$ , and  $(c, F)$ . Then

$$\begin{array}{rclcl} d-e+F-G & -U & =m+2, \\ -d+e+F-G & -V & =m+2, \\ b-c & +F-G & +T & =m-2, \\ -b+c & +F-G+S & & =m-2, \end{array}$$

But  $-S-T-U-V=8$  contradicts Lemma 1. Q.E.D.

We now study the global structure of an irreducible component  $Q$  of the vertex-labelable poset  $P$ . Let  $q$  denote the number of elements of  $Q$ , let  $T$  denote the basic tree of  $Q$ , and let  $n$  denote the number of elements of  $T$ . The number  $n$  could be called the *rank* of  $Q$ , since it will be seen to be analogous to the rank of an irreducible Weyl group or the rank of a simple Lie algebra.

**LEMMA 6.** *The basic tree of  $Q$  has exactly one root and is either a chain or “Y-shaped”, i.e., it has at most one vertex with three or more adjacent vertices.*

*Proof.* Lemma 3 precludes the existence of more than one minimal element of  $Q$ . If there is more than one “branching” in  $T$ , use Lemma 3 to produce a vertex in the basic tree which is covered by three or more elements, contradicting Lemma 4.

Now set  $n = b + c + d + 1$  where  $b$  is the number of vertices in the branch of the basic tree  $T$  containing the root ( $b = 0$  if the root is covered by two elements), and  $c$  and  $d$  are the numbers of elements in the other two branches of  $T$ . Refer to the elements of  $T$  with the letters shown in Fig. 5.

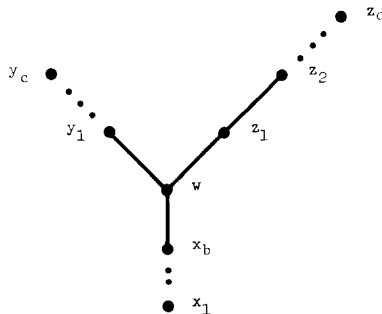


FIG. 5. Basic tree of an irreducible component.

LEMMA 7. *The following connected Dynkin diagrams with special nodes are the only possibilities for the basic tree of the irreducible component  $Q$ :  $\mathbf{A}_n[j]$ ,  $1 \leq j \leq n$ ,  $\mathbf{D}_n[1]$ ,  $\mathbf{D}_n[n-1]$ ,  $\mathbf{D}_n[n]$ ,  $\mathbf{E}_6[1]$ ,  $\mathbf{E}_6[2]$ ,  $\mathbf{E}_6[6]$ ,  $\mathbf{E}_7[1]$ ,  $\mathbf{E}_7[2]$ ,  $\mathbf{E}_7[7]$ ,  $\mathbf{E}_8[1]$ ,  $\mathbf{E}_8[2]$ , and  $\mathbf{E}_8[8]$ .*

*Proof.* Let  $s$  equal  $p$  minus the sum of the labels of the minimal elements of  $P$  lying outside  $Q$ . Consider the empty ideal of  $P$  together with the  $n$  ideals of  $P$  each generated by one element of the basic tree  $T$  of  $Q$ . The following system of  $n+1$  equation in  $n+1$  unknowns is obtained:

$$\begin{array}{rcl}
 -x_1 & & +s=0 \\
 x_1-x_2 & & +s=2 \\
 & \ddots & \vdots \\
 & x_b-w & +s=2 \\
 & w-y_1 & -z_1 & +s=2(b+1) \\
 & y_1-y_2 & -z_1 & +s=2(b+2) \\
 & & \ddots & \vdots \\
 & & y_c-z_1 & +s=2(b+c+1) \\
 & -y_1 & +z_1-z_2 & +s=2(b+2) \\
 & & \ddots & \vdots \\
 & -y_1 & & +z_d+s=2(b+d+1).
 \end{array}$$

The unique solution is

$$\begin{array}{ll}
 \text{For } 1 \leq i \leq b, & x_i = i(s-i+1), \\
 & w = (b+1)(s-b), \\
 \text{for } 1 \leq j \leq c, & y_j = (b+j+1)(s-b-j) - jz_1, \\
 \text{for } 1 \leq k \leq d, & z_k = (b+k+1)(s-b-k) - ky_1,
 \end{array}$$

and

$$s = \frac{-b^2cd + bcd + c^2d + cd^2 + b^2 + c^2 + d^2 + 2bc + 2bd + 4cd + 3b + 3c + 3d + 2}{-bcd + b + c + d + 2}.$$

Since  $w = (b+1)(s-b)$ , this vertex label will be negative if  $s-b < 0$ :

$$s-b = \frac{bcd + c^2d + cd^2 + bc + bd + 4cd + c^2 + d^2 + b + 3c + 3d + 2}{-bcd + b + c + d + 2}.$$

It is easy to check that the denominator of this expression is positive only for the following unordered values of  $b$ ,  $c$ , and  $d$ :  $\{\{0, j, k\} : 0 \leq j < \infty, 0 \leq k < \infty\} \cup \{\{1, 1, k\} : 1 \leq k < \infty\} \cup \{\{1, 2, 2\}, \{1, 2, 3\}, \{1, 2, 4\}\}$ . Lemma 1 thus implies that no other values are permissible. Consult Fig. 1.

These rooted trees are exactly the Dynkin diagrams with special node listed in the statement of the lemma.

The next lemma uses direct constructions to determine which of these possibilities are actually basic trees for components of vertex-labelable posets.

**LEMMA 8.** *Each of the basic trees  $\mathbf{A}_n[j]$ ,  $\mathbf{D}_n[1]$ ,  $\mathbf{D}_n[n]$ ,  $\mathbf{E}_6[6]$ , and  $\mathbf{E}_7[7]$  determines one possible irreducible component of a vertex-labelable poset with a unique vertex labeling. None of the rooted Dynkin diagrams  $\mathbf{E}_6[2]$ ,  $\mathbf{E}_7[1]$ ,  $\mathbf{E}_7[2]$ ,  $\mathbf{E}_8[8]$ ,  $\mathbf{E}_8[1]$ , or  $\mathbf{E}_8[2]$  is a basic tree for an irreducible component of a vertex-labelable poset.*

*Proof.* If elements  $b$  and  $c$  both cover  $d$ , and  $e$  is the unique element required by Lemma 5 which covers both  $b$  and  $c$ , then the proof of Lemma 3 implies that  $\pi(e) = \pi(d)$ . This fact, Lemma 4, and Lemma 5 will be collectively referred to with the phrase "local structure." Let  $s$  be as in the previous proof. Note that  $s = x_1$ , the label of the minimal element of the component at hand.

First consider  $\mathbf{E}_6[2]$ ,  $\mathbf{E}_7[2]$ , and  $\mathbf{E}_8[2]$ . Let  $v$  be the unique element covering both  $y_1$  and  $z_1$ . By considering the ideals  $(v)$  and  $(y_1, z_1)$ , one obtains  $v = (y_1 + z_1)/2 + 1$ . Computing  $v$  for these three cases yields the numbers  $v = 31, \frac{143}{2}$ , and 202. But local structure implies that  $v = w = 42, 96$ , and 270.

Now consider  $\mathbf{E}_7[1]$ . After computing the values for the basic tree and applying local structure, one can immediately construct as much of the irreducible component  $Q$  as is shown in the first part of Fig. 6. Using the ideal  $(96')$ , one finds  $c = 66$ . Then the ideal  $(52')$  leads to  $d = 0$ , implying that  $52'$  is not covered by such an element. The second part of Fig. 6 shows the situation now. Using  $(66')$ , one computes  $e = 34$ . Considering the ideal

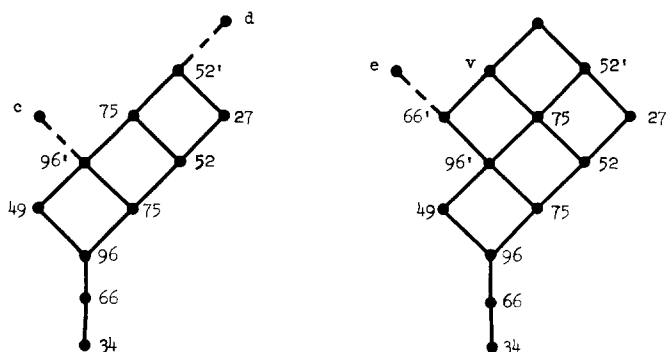
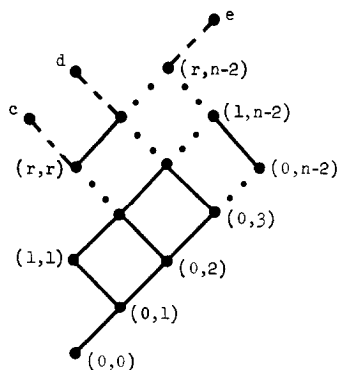


FIG. 6. Lemma  $E_7[1]$ .

FIG. 7. Lemma 8:  $D_n[n]$ .

( $v$ ) leads to  $v=47$ . But  $v=96$  by local structure. Similar arguments lead to inconsistencies in the 6th and 12th ranks of the irreducible components of  $\mathbf{E}_8[1]$  and  $\mathbf{E}_8[8]$ .

Next consider  $\mathbf{D}_n[n]$  with  $n \geq 4$ . Denote the elements of  $Q$  as shown in Fig. 7, and look up the labels for the basic tree in the proof of Lemma 7. Use induction on  $r$ . Assume that  $\pi(i.i) = x_1$  for  $i \leq r$ . First consider the ideal  $(r, r)$ :

$$\begin{aligned} x_1 - c - z_r + s &= (r+1)(r+2), \\ c &= 2x_1 - z_r - (r+1)(r+2), \\ c &= 0. \end{aligned}$$

Next consider the ideal  $(r, r+1)$ :

$$\begin{aligned} w - d - z_{r+1} + s &= (r+1)(r+4), \\ d &= w - z_{r+1} + x_1 - (r+1)(r+4), \\ d &= x_1. \end{aligned}$$

And consider the ideal  $(r, n-2)$  for  $r \leq n-4$ :

$$\begin{aligned} z_{n-r-3} - e - x_1 + s &= r(r+1) - 2(r+1)(n-1), \\ e &= z_{n-r-3} + 2(r+1)(n-1) - r(r+1), \\ e &= 0. \end{aligned}$$

After consideration of the ideals  $(n-3, n-2)$  and  $(n-2, n-2)$ , one can conclude that  $Q$  has  $q = n(n-1)/2 = s$  elements arranged as in Fig. 2.

The construction of  $Q$  for  $\mathbf{A}_n[j]$ ,  $\mathbf{D}_n[1]$ ,  $\mathbf{E}_6[6]$ , and  $\mathbf{E}_7[7]$  are similar

and will be omitted. In each case one finds that  $s = q$  and that the labeled component  $Q$  constructed is as shown in Fig. 2.

The explicit verification of *all* linear conditions for case  $\mathbf{A}_n[j]$  is performed in [Pr3]. The computations for case  $\mathbf{D}_n[n]$  are similar. All linear conditions in the other three cases are easily verified during the construction of  $Q$ . The proof of Lemma 8 is complete.

*Proof of Theorem 2.* Since  $s = q$  in each good case of Lemma 8, each possible irreducible component is in fact a vertex-labelable poset by itself. Direct computations with the  $J$  operator confirm that the description given in the statement of Theorem 2 agrees with the posets constructed in the proof of Lemma 8. It is obvious that the direct sum of vertex-labelable posets is a vertex-labelable poset. Since Lemma 8 lists the desired possibilities for components of reducible vertex-labelable posets, the third assertion of Theorem 2 is also true.

*Proof of Corollary.* It is conceivable that  $P = Q_1 \oplus Q_2$ , with the following equation holding for every ideal  $I_1 \subseteq Q_1$ :

$$\sum_{\substack{x \text{ maximum} \\ \text{in } I_1}} \pi(x) - |I_1| - \sum_{\substack{y \text{ minimum} \\ \text{in } Q_1 - I_1}} \pi(y) + |Q_1 - I_1| = \alpha_1,$$

where  $\alpha_1 \neq 0$ , and with a similar equation holding for every ideal  $I_2 \subseteq Q_2$ . If  $\alpha_2 = -\alpha_1$ , then  $P$  is vertex-labelable. This kind of situation is ruled out by the proof of Lemma 8, which shows that  $\alpha_i = 0$  for every possible irreducible component  $Q_i$  of a vertex-labelable poset.

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